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Ramanujan-type continuous measures for biorthogonal q-rational functions

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Abstract. It is shown that Ramanujan-type continuous measures for a hierarchy of ${}_4\phi_3$ biorthogonal rational functions can be systematically built from simple cases of the Rogers–Szegö (for $0 < q < 1$) and Stieltjes–Wigert (for $q > 1$) polynomials by using the Berg–Ismail procedure of attaching generating functions to measures.

1. Introduction

The idea of extending non-negative real weight functions to more general complex ones goes back at least to Szegö [1] and Hahn [2]. The goal is to obtain orthogonality relations with continuous measures for classical polynomials in a wider range of parameters. The detailed motivation for the study of such extensions can be found in more recent publications [3–7] on this subject. [6] also contains examples of complex weight functions with respect to which the Jacobi, Laguerre, little q-Jacobi and Askey–Wilson polynomials are orthogonal.

Recently it has become clear that there exists a natural way of constructing the particular type of complex measures for an entire hierarchy of classical q-polynomials, ranging from the one-parameter continuous q-Hermite polynomials [8] to the five-parameter Askey–Wilson polynomials [9]. It turns out that the use of the modular and periodicity properties of the theta-functions $\vartheta_i(z, q)$, $i = 1, 2, 3, 4$ [10, 11], which enter the standard weight functions for all of these polynomials, leads directly to complex weight functions with an infinite support [12–17]. The advantage of the Ramanujan-type measures thus obtained is that they admit the transformation $q \to q^{-1}$.

As is shown in [18], one can also arrive at the same results for a hierarchy of the Askey–Wilson polynomials by using the Berg and Ismail procedure of attaching generating functions to orthogonality measures [19]. The purpose of the present paper is to apply the same technique of constructing Ramanujan-type measures to a family of ${}_4\phi_3$ biorthogonal rational functions [7], which are the unit circle analogues of the Askey–Wilson polynomials.

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2. The Rogers–Szegö ladder $(0 < q < 1)$

The q-analogue of Hermite polynomials on the unit circle are the Rogers–Szegö polynomials [1, 20, 21]

$$H_n(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k = {}_2\phi_0(q^{-n}, 0; q, zq^n) \tag{2.1}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

and $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ is the q-shifted factorial. The basic hypergeometric function ${}_2\phi_0$ in (2.1) represents a particular case $(r = 2, s = 0, a_1 = q^{-n}, a_2 = 0)$ of the general definition

$${}_r\phi_s \begin{bmatrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{bmatrix}; q, z \Bigg] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} [(-1)^n q n(n-1)/2]^{1+s-r} z^n \tag{2.2}$$

with the convention $(a_1, \ldots, a_r; q)_n = \prod_{j=1}^{r}(a_j;q)_n$ [22].

Szegö [1] has proved that the polynomials (2.1) satisfy the orthogonality relation

$$\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} H_m(-q^{-1/2}z^*;q) H_n(-q^{-1/2}z;q) \vartheta_3\left(\frac{\log z}{2\mathrm{i}}, q^{1/2}\right) \frac{\mathrm{d}z}{z} = \frac{(q;q)_m}{q^m}\delta_{mn} \tag{2.3}$$

where $\vartheta_3(z,q)$ is the theta-function, i.e.

$$\vartheta_3(z,q) \equiv \vartheta_3(z|\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} \mathrm{e}^{2\mathrm{i}kz} \tag{2.4}$$

and $q = \exp(\pi \mathrm{i}\tau)$ [10]. The orthogonality relation (2.3) on the unit circle can be transformed into the Ramanujan-type orthogonality of the form [14]

$$\int_{-\infty}^{\infty} H_m(-q^{-1/2}\,\mathrm{e}^{2\mathrm{i}\kappa x};q) H_n(-q^{-1/2}\,\mathrm{e}^{-2\mathrm{i}\kappa x};q)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,\frac{(q;q)_n}{q^n}\delta_{mn}$$
$$q = \exp(-2\kappa^2). \tag{2.5}$$

Once the relation (2.5) is established, it is readily verified by a direct evaluation with the aid of the explicit representation (2.1), the Fourier transformation

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2+2\mathrm{i}xy}\,\mathrm{d}x = \sqrt{\pi}\,\mathrm{e}^{-y^2} \tag{2.6}$$

and Gauss identity (see, for example, [22])

$$(z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q qk(k-1)/2(-z)^k. \tag{2.7}$$

It follows from (2.1) and the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{2.8}$$

that the generating function for the Rogers–Szegö polynomials has the form [21]

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(x;q) = e_q(t) e_q(xt) \tag{2.9}$$

where $e_q(z)$ is the q-exponential function, i.e.

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = (z;q)^{-1} \qquad |z| < 1. \tag{2.10}$$

Hence, combining (2.9) with the orthogonality relation (2.5) leads to the particular case

$$\int_{-\infty}^{\infty} e_q(-t\, \mathrm{e}^{2\mathrm{i}\kappa x}) e_q(-\tau\, \mathrm{e}^{-2\mathrm{i}\kappa x})\, \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}\, e_q(t\tau) E_q(-q^{1/2}t) E_q(-q^{1/2}\tau) \tag{2.11}$$

of a Ramanujan's q-extension of Cauchy form of the beta integral [23, 24, 4], namely

$$\int_{-\infty}^{\infty} e_q(\alpha\, \mathrm{e}^{2\mathrm{i}\kappa x}) e_q(\beta\, \mathrm{e}^{-2\mathrm{i}\kappa x})\, \mathrm{e}^{-x^2+2mx}\, \mathrm{d}x = \sqrt{\pi}\, \mathrm{e}^{m^2} e_q(\alpha\beta) E_q(\alpha q^{1/2}\, \mathrm{e}^{2\mathrm{i}m\kappa}) E_q(\beta q^{1/2}\, \mathrm{e}^{-2\mathrm{i}m\kappa}). \tag{2.12}$$

Two points demand comments, however. First, it is common to express (see, for example, [22–24, 4]) two integrals of Ramanujan, i.e. (2.12) and a q-extension of Euler's beta integral

$$\int_{-\infty}^{\infty} E_q(\alpha q^{1/2}\, \mathrm{e}^{2\kappa x}) E_q(\beta q^{1/2}\, \mathrm{e}^{-2\kappa x})\, \mathrm{e}^{-x^2+2mx}\, \mathrm{d}x = \sqrt{\pi}\, \mathrm{e}^{m^2} E_q(-\alpha\beta) e_q(\alpha\, \mathrm{e}^{2m\kappa}) e_q(\beta\, \mathrm{e}^{-2m\kappa}) \tag{2.13}$$

in terms of the infinite product $(z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n)$, rather than the q-exponential function (2.10) and its reciprocal

$$E_q(z) := \sum_{n=0}^{\infty} \frac{qn(n-1)/2}{(q;q)_n} z^n = (-z;q)_\infty. \tag{2.14}$$

We find it more appropriate to use these forms (2.12) and (2.13) (cf [17, 18]). Second, since the term $2mx$ in the argument of the exponential function $\exp(-x^2+2mx)$ can be removed by the translation $x \to x+m$ and redefinition of the parameters α and β, to simplify a proof of (2.12) and (2.13) it is usually assumed that $m = 0$. Thus the parameter m in (2.12) and (2.13) is supposed to be real. In fact, these formulae are also valid for arbitrary complex m [25]. This can be proved by a direct evaluation of integrals (2.12) and (2.13) with the aid of the formulae (2.6), (2.8) and

$$\left(-\frac{q}{z};q\right)_n E_q(z) = qn(n+1)/2 z^{-n} E_q(zq^{-n}). \tag{2.15}$$

As the real part of a complex number m can be recovered by the shift $x \to x - \mathrm{Re}\, m$, without loss of generality it may be assumed that m is imaginary, i.e. $m = \mathrm{i}y$. Consequently, the representations (2.12) and (2.13) are also instances of Fourier transformations and this circumstance will be used in what follows.

Now we return to the integrand in (2.11), which represents a weight function for the biorthogonal polynomials

$$\begin{aligned} p_n(z;t,\tau;q) &:= (q^{-1/2}\tau;q)_n\, {}_2\phi_1(q^{-n}, q^{1/2}t; q^{3/2-n}/\tau; q, qz/\tau) \\ &= \frac{(t\tau;q)_n}{(q^{1/2}t)^n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, tz, q^{1/2}t \\ t\tau, 0 \end{matrix}; q, q\right] \end{aligned} \tag{2.16}$$

studied by Pastro [4]. Observe that the second line in (2.16) follows from the Jackson transformation formula [22]

$${}_2\phi_1(q^{-n}, b; c; q, z) = \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bz/cq^n \\ bq^{1-n}/c, 0 \end{matrix}; q, q\right] \tag{2.17}$$

for terminating basic hypergeometric series ${}_2\phi_1$, upon employing the relation

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\left(\frac{a}{b}\right)^n. \tag{2.18}$$

The normalization of polynomials (2.16) is chosen in such a way, that when $t = \tau = 0$ they coincide with the Rogers–Szegö polynomials (2.1), i.e.

$$p_n(z;0,0;q) = H_n(q^{-1/2}z;q). \tag{2.19}$$

This formula follows from the readily verified limit relation

$$\lim_{c\to\infty} {}_2\phi_1(q^{-n},a;c;q,cz) = {}_2\phi_0(q^{-n},a;q,z). \tag{2.20}$$

An explicit evaluation of the Ramanujan-type biorthogonality integral

$$\int_{-\infty}^{\infty} p_m(-\mathrm{e}^{2\mathrm{i}\kappa x};t,\tau;q)p_n(-\mathrm{e}^{-2\mathrm{i}\kappa x};\tau,t;q)e_q(-t\,\mathrm{e}^{2\mathrm{i}\kappa x})e_q(-\tau\,\mathrm{e}^{-2\mathrm{i}\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x$$
$$= \sqrt{\pi}\,\frac{(q;q)_n}{q^n}e_q(q^n t\tau)E_q(-q^{1/2}t)E_q(-q^{1/2}\tau)\delta_{mn} \tag{2.21}$$

is given in the appendix.

Multiplying both sides of (2.21) by a constant factor $t_3^m t_4^n (q;q)_m^{-1}(q;q)_n^{-1}$ and summing over indices m and n from zero to infinity with the aid of the generating function [7]

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}p_n(z;a,b;q) = \frac{(atz,q^{-1/2}bt;q)_\infty}{(q^{-1/2}tz,t;q)_\infty} = \frac{e_q(t)e_q(q^{-1/2}tz)}{e_q(q^{-1/2}bt)e_q(atz)} \tag{2.22}$$

for the polynomials (2.16), we get ($t = t_1$, $\tau = t_2$)

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}E_q(q^{1/2}t_1t_3\,\mathrm{e}^{2\mathrm{i}\kappa x})E_q(q^{1/2}t_2t_4\,\mathrm{e}^{-2\mathrm{i}\kappa x})\prod_{j=1}^{4}e_q(-t_j\,\mathrm{e}^{(-)^{j+1}2\mathrm{i}\kappa x})\,\mathrm{d}x$$
$$= \sqrt{\pi}\,e_q(t_1t_2)e_q(t_2t_3)e_q(t_3t_4)e_q(t_1t_4)E_q(-t_1t_2t_3t_4)\prod_{j=1}^{4}E_q(-q^{1/2}t_j). \tag{2.23}$$

The integrand in (2.23) represents a weight function for the biorthogonality relation

$$\int_{-\infty}^{\infty} r_m(-\mathrm{e}^{2\mathrm{i}\kappa x};t_1,t_2,t_3,t_4;q)r_n(-\mathrm{e}^{-2\mathrm{i}\kappa x};t_2,t_1,t_4,t_3;q)$$
$$\times E_q(q^{1/2}t_1t_3\,\mathrm{e}^{2\mathrm{i}\kappa x})E_q(q^{1/2}t_2t_4\,\mathrm{e}^{-2\mathrm{i}\kappa x})\mathrm{e}^{-x^2}\prod_{j=1}^{4}e_q(-t_j\,\mathrm{e}^{(-)^{j+1}2\mathrm{i}\kappa x})\,\mathrm{d}x$$
$$= \sqrt{\pi}\,\frac{(t_1t_2,t_3t_4,t_1t_2t_3t_4q^{n-1},q;q)_n}{q^n(t_1t_2t_3t_4;q)_{2n}}\delta_{mn}e_q(t_1t_2)e_q(t_2t_3)e_q(t_3t_4)e_q(t_1t_4)$$
$$\times E_q(-t_1t_2t_3t_4)\prod_{l=1}^{4}E_q(-q^{1/2}t_l) \tag{2.24}$$

which is satisfied by ${}_4\phi_3$ rational functions of Al-Salam and Ismail [7]

$$r_n(z;t_1,t_2,t_3,t_4;q) = \frac{(t_1t_2;q)_n}{(q^{1/2}t_1)^n}\,{}_4\phi_3\left[\begin{matrix}q^{-n},t_1z,q^{1/2}t_1,t_1t_2t_3t_4q^{n-1}\\ q^{1/2}t_1t_3z,t_1t_2,t_1t_4\end{matrix};q,q\right]. \tag{2.25}$$

The Ramanujan-type biorthogonality relation (2.24) was derived by a different method in [16], but it can also be directly evaluated in exactly the same way as the integral (2.21) (see the appendix), namely, by using the definition (2.25), the representation (2.23) and the Pfaff–Saalschütz (A.7) and Chu–Vandermonde (A.12) q-sums. The normalization in (2.25) is chosen in such a way that for $t_3 = t_4 = 0$ these rational functions and the biorthogonality relation (2.24) reduce to the biorthogonal polynomials (2.16) and the relation (2.21), respectively. Consequently, for $t_j = 0$, $1 \leqslant j \leqslant 4$, the rational functions (2.25) coincide with the Rogers–Szegö polynomials (2.1), satisfying the Ramanujan-type orthogonality relation (2.5).

3. The Stieltjes–Wigert ladder $(q > 1)$

The Stieltjes–Wigert polynomials [22]

$$S_n(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} z^k = {}_1\phi_1(q^{-n};0;q,zq^{n+1}) \tag{3.1}$$

represent the lower level in another ladder of ${}_4\phi_3$ biorthogonal rational functions [7], corresponding to the values $q > 1$ of the parameter q. They are related to the Rogers–Szegö polynomials (2.1) by the Fourier transformation [14]

$$S_n(\alpha\,\mathrm{e}^{2\kappa x};q)\,\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(\alpha\,\mathrm{e}^{-2\mathrm{i}\kappa y};q)\,\mathrm{e}^{\mathrm{i}xy-y^2/2}\,\mathrm{d}y \tag{3.2}$$

where α is an arbitrary complex number. It is interesting to compare (3.2) with the relation

$$S_n(x;q) = \frac{1}{\pi} \int_0^{\pi} H_n(x\,\mathrm{e}^{\mathrm{i}\varphi};q)\vartheta_3(\varphi,q)\,\mathrm{d}\varphi \tag{3.3}$$

derived by Carlitz in [21].

From (2.5) and (3.2) it follows that the Stieltjes–Wigert polynomials (3.1) satisfy the Ramanujan-type orthogonality relation of the form [14]

$$\int_{-\infty}^{\infty} S_m(-q^{-1/2}\,\mathrm{e}^{2\kappa x};q) S_n(-q^{-1/2}\,\mathrm{e}^{2\kappa x};q)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,\frac{(q;q)_m}{q^m}\,\delta_{mn}. \tag{3.4}$$

This can also be readily verified by a direct evaluation of (3.4) with the aid of the explicit representation (3.1), the formula (cf (2.6))

$$\int_{-\infty}^{\infty} \mathrm{e}^{2xy-x^2}\,\mathrm{d}x = \sqrt{\pi}\,\mathrm{e}^{y^2} \tag{3.5}$$

and the identity (2.7).

As in the previous section for the Rogers–Szegö ladder we could have started with the orthogonality relation (3.4) and the generating function [26]

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} S_n(z;q) = e_q(t)\,{}_0\phi_1(0;q,qzt) \tag{3.6}$$

for the Stieltjes–Wigert polynomials (3.1). However, for the approach under consideration, it is important that generating functions are expressed in terms of the q-exponential functions (2.10) and (2.14), because biorthogonality relations at every next level follow from their properties (2.15) and (A.5). Fortunately, there exists along with (3.4) another orthogonality relation (cf [21])

$$\int_{-\infty}^{\infty} S_m(-q^{1/2-m}\,\mathrm{e}^{2\kappa x};q) S_n(-q^{1/2-n}\,\mathrm{e}^{-2\kappa x};q)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}(-1)^n q^{-n(n-1)/2}(q;q)_n\delta_{mn} \tag{3.7}$$

since the Stieltjes and Hamburger moment problems [27] for the Stieltjes–Wigert polynomials (3.1) are indeterminate. This is a direct consequence of (2.5) and the identity

$$H_n(z; q^{-1}) = S_n(q^{-n}z; q) \tag{3.8}$$

which follows from the definitions (2.1) and (3.1) upon using the transformation property of the q-binomial coefficients [22]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{3.9}$$

In analogy with the q^{-1}-Hermite polynomials $h_n(x|q) = \mathrm{i}^{-n} H_n(\mathrm{i}x|q^{-1})$ [28] it is convenient to consider Stieltjes–Wigert polynomials of the form

$$s_n(z; q) = \mathrm{i}^{-n} H_n(z; q^{-1}) = \mathrm{i}^{-n} S_n(q^{-n}z; q). \tag{3.10}$$

To distinguish them from the standard Stieltjes–Wigert polynomials (3.1), they will be denoted by the small letter s. From (3.7) it is evident that

$$\int_{-\infty}^{\infty} s_m(-q^{1/2}\,\mathrm{e}^{2\kappa x}; q) s_n(-q^{1/2}\,\mathrm{e}^{-2\kappa x}; q)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\, q^{-n(n-1)/2}(q; q)_n \delta_{mn}. \tag{3.11}$$

A generating function for the polynomials (3.10) is [21]

$$\sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{(q; q)_n} q^{n(n-1)/2} s_n(z; q) = E_q(-t) E_q(-zt). \tag{3.12}$$

This can be easily checked by using Cauchy's multiplication rule $\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} b_k$ for the product of two sequences $\{a_n\}$ and $\{b_k\}$ and the definition of the q-exponential function (2.14). Combining (3.12) with (3.11) leads to the integral

$$\int_{-\infty}^{\infty} E_q(q^{1/2} t\,\mathrm{e}^{2\kappa x}) E_q(q^{1/2}\tau\,\mathrm{e}^{-2\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\, e_q(t) e_q(\tau) E_q(-t\tau) \tag{3.13}$$

which is a particular case of Ramanujan's identity (2.13) with the vanishing parameter m.

The integrand in (3.13) represents a weight function for the biorthogonal polynomials

$$\begin{aligned} \tilde{p}_n(z; t, \tau; q) &= \mathrm{i}^{-n} p_n(z; t, \tau; q^{-1}) \\ &= \mathrm{i}^{-n} (q^{1/2}\tau; q^{-1})_n\, {}_2\phi_1(q^{-n}, q^{1/2}t^{-1}; \tau q^{3/2-n}; q, qtz) \end{aligned} \tag{3.14}$$

where $p_n(z; t, \tau; q)$ are defined by (2.16). Indeed, it is not difficult to verify by a direct evaluation (actually repeating the same line of reasoning *mutatis mutandis* as for the case of $0 < q < 1$, considered in the appendix) that the following biorthogonality relation holds

$$\begin{aligned} \int_{-\infty}^{\infty} & \tilde{p}_m(-\mathrm{e}^{2\kappa x}; t, \tau; q) \tilde{p}_n(-\mathrm{e}^{-2\kappa x}; \tau, t; q) E_q(qt\,\mathrm{e}^{2\kappa x}) E_q(q\tau\,\mathrm{e}^{-2\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x \\ &= \sqrt{\pi}\, q^{-n(n-1)/2}(q; q)_n E_q(-q^{1-n}t\tau) e_q(q^{1/2}t) e_q(q^{1/2}\tau) \delta_{mn}. \end{aligned} \tag{3.15}$$

Observe that for $t = \tau = 0$ the polynomials (3.14) reduce to the Stieltjes–Wigert polynomials (3.10), i.e.

$$\tilde{p}_n(z; 0, 0; q) = s_n(q^{1/2}z; q). \tag{3.16}$$

This follows from (3.14), (3.1) and (3.10) upon using the limit relation

$$\lim_{a\to\infty} {}_2\phi_1(q^{-n}, a; c; q, z/a) = {}_1\phi_1(q^{-n}; c; q, z). \tag{3.17}$$

A generating function for the polynomials (3.14)

$$\sum_{n=0}^{\infty} \frac{(-\mathrm{i}\alpha)^n}{(q;q)_n} q^{n(n-1)/2} \tilde{p}_n(z;t,\tau;q) = \frac{e_q(q^{1/2}\alpha\tau)e_q(\alpha t z)}{e_q(\alpha)e_q(q^{1/2}\alpha z)} \tag{3.18}$$

can be obtained by a direct evaluation of the left member of (3.18) with the aid of the q-binomial theorem (2.8). The biorthogonality relation (3.15) and the generating function (3.18) imply the integral representation

$$\int_{-\infty}^{\infty} e_q(-q^{1/2}t_1t_3\,\mathrm{e}^{2\kappa x})e_q(-q^{1/2}t_2t_4\,\mathrm{e}^{-2\kappa x}) \prod_{j=1}^{4} E_q(qt_j\,\mathrm{e}^{(-)^{j+1}2\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x$$
$$= \sqrt{\pi}\,E_q(-qt_1t_2)E_q(-qt_2t_3)E_q(-qt_3t_4)E_q(-qt_1t_4)e_q(qt_1t_2t_3t_4)\prod_{k=1}^{4} e_q(q^{1/2}t_k). \tag{3.19}$$

The integrand in (3.19) represents a weight function for the biorthogonality relation [16]

$$\int_{-\infty}^{\infty} \tilde{r}_m(-\mathrm{e}^{2\kappa x};t_1,t_2,t_3,t_4;q)\tilde{r}_n(-\mathrm{e}^{-2\kappa x};t_2,t_1,t_4,t_3;q)$$
$$\times e_q(-q^{1/2}t_1t_3\,\mathrm{e}^{2\kappa x})e_q(-q^{1/2}t_2t_4\,\mathrm{e}^{-2\kappa x}) \prod_{j=1}^{4} E_q(qt_j\,\mathrm{e}^{(-)^{j+1}2\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x$$
$$= \frac{\sqrt{\pi}(q;q)_n}{1-Tq^{1-2n}} q^{-n(n-1)/2} E_q(-q^{1-n}t_1t_2)E_q(-q^{1-n}t_3t_4)E_q(-qt_1t_4)E_q(-qt_2t_3)$$
$$\times e_q(q^{2-n}T)\prod_{k=1}^{4} e_q(q^{1/2}t_j) \tag{3.20}$$

where $T \equiv t_1t_2t_3t_4 \neq q^k$, k is an arbitrary integer number, and q^{-1}-rational functions $\tilde{r}_m(z)$ and $\tilde{r}_n(z)$ of Al-Salam and Ismail are defined as (cf (2.25))

$$\tilde{r}_n(z;t_1,t_2,t_3,t_4;q) = \mathrm{i}^{-n} r_n(z;t_1,t_2,t_3,t_4;q^{-1})$$
$$= (\mathrm{i}q)^n q^{-n^2/2} \frac{(\tilde{t}_1\tilde{t}_2;q)}{\tilde{t}_2^n}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, \tilde{t}_1z^{-1}, q^{1/2}\tilde{t}_1, \tilde{T}q^{n-1} \\ q^{1/2}\tilde{t}_1\tilde{t}_3z^{-1}\tilde{t}_1\tilde{t}_2, \tilde{t}_1\tilde{t}_4 \end{matrix}; q, q\right] \tag{3.21}$$

with $\tilde{t}_j = t_j^{-1}$, $j = 1, 2, 3, 4$, and $\tilde{T} = T^{-1}$. The biorthogonality relation (3.20) can be directly evaluated by using the definition (3.21) and the formulae (2.15), (A.5), (A.7) and (A.11).

4. Concluding remarks

As has been mentioned in section 2, the Rogers–Szegö (2.1) and the Stieltjes–Wigert (3.1) polynomials, which represent the ground levels of two ${}_4\phi_3$ ladders for different values $0 < q < 1$ and $q > 1$ of the parameter q, are related to each other by the Fourier transformation (3.2). An interesting direction for further study is to extend this transformation to the case of the higher levels, i.e. to find an integral relation between ${}_4\phi_3$ biorthogonal rational functions (2.25) and (3.21).

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Appendix

To evaluate the Ramanujan-type biorthogonality integral

$$I_{mn}(t,\tau;q)=\int_{-\infty}^{\infty} p_m(-\mathrm{e}^{2\mathrm{i}\kappa x};t,\tau;q)\,p_n(-\mathrm{e}^{-2\mathrm{i}\kappa x};\tau,t;q)\,e_q(-t\,\mathrm{e}^{2\mathrm{i}\kappa x})\,e_q(-\tau\,\mathrm{e}^{-2\mathrm{i}\kappa x})\,\mathrm{e}^{-x^2}\,\mathrm{d}x \tag{A.1}$$

substitute the explicit expression

$$p_m(-\mathrm{e}^{2\mathrm{i}\kappa x};t,\tau;q)=(q^{-1/2}\tau;q)_m\sum_{j=0}^{m}\frac{(q^{-m},q^{1/2}t;q)_j}{(q,q^{3/2-m}/\tau;q)_j}\left(-\frac{q}{\tau}\,\mathrm{e}^{2\mathrm{i}\kappa x}\right)^j \tag{A.2}$$

into (A.1) for both polynomials in it, which are defined by (2.16). This gives

$$\begin{aligned}I_{mn}(t,\tau;q)&=(q^{-1/2}\tau;q)_m(q^{-1/2}t;q)_n\\&\quad\times\sum_{j=0}^{m}\frac{(q^{-m},q^{1/2}t;q)_j}{(q,q^{3/2-m}/\tau;q)_j}\left(-\frac{q}{\tau}\right)^j\sum_{k=0}^{n}\frac{(q^{-n},q^{1/2}\tau;q)_k}{(q,q^{3/2-n}/t;q)_k}\left(-\frac{q}{t}\right)^k\\&\quad\times\int_{-\infty}^{\infty}\mathrm{e}^{2\mathrm{i}\kappa(j-k)x-x^2}e_q(-t\,\mathrm{e}^{2\mathrm{i}\kappa x})e_q(-\tau\,\mathrm{e}^{-2\mathrm{i}\kappa x})\,\mathrm{d}x.\end{aligned} \tag{A.3}$$

The integration over the variable x in (A.3) is performed by (2.12) with $m=2\mathrm{i}\kappa(j-k)$, $\alpha=-t$ and $\beta=-\tau$, i.e.

$$\begin{aligned}I_{mn}(t\tau;q)&=\sqrt{\pi}\,e_q(t\tau)(q^{-1/2}\tau;q)_m(q^{-1/2}t;q)_n\sum_{j=0}^{m}\frac{(q^{-m},q^{1/2}t;q)_j}{(q,q^{3/2-m}/\tau;q)_j}\left(-\frac{q}{\tau}\right)^j\\&\quad\times\sum_{k=0}^{n}\frac{(q^{-n},q^{1/2}\tau;q)_k}{(q,q^{3/2-n}/t;q)_k}\left(-\frac{q}{t}\right)^k q^{(j-k)^2/2}E_q(-tq^{j-k+1/2})E_q(-\tau q^{k-j+1/2}).\end{aligned} \tag{A.4}$$

The formulae (2.15) and

$$E_q(z)=(-z;q)_kE_q(zq^k) \tag{A.5}$$

enable the right-hand side of (A.4) to be written as

$$\begin{aligned}I_{mn}(t\tau;q)&=\sqrt{\pi}\,e_q(t\tau)E_q(-q^{1/2}t)E_q(-q^{1/2}\tau)(q^{-1/2}\tau;q)_m(q^{-1/2}t;q)_n\\&\quad\times\sum_{j=0}^{m}\frac{(q^{-m},q^{1/2}/\tau;q)_j}{(q,q^{3/2-m}/\tau;q)_j}q^j\sum_{k=0}^{n}\frac{(q^{-n},q^{1/2}\tau,q^{1/2-j}/t;q)_k}{(q,q^{1/2-j}\tau,q^{3/2-n}/t;q)_k}q^k.\end{aligned} \tag{A.6}$$

The sum with respect to the index k in (A.6) is a particular case of the Pfaff–Saalschütz q-sum

$${}_3\phi_2\left[\begin{matrix}q^{-n},a,b\\c,abc^{-1}q^{1-n}\end{matrix};q,q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n} \tag{A.7}$$

with $a = q^{1/2}\tau$, $b = q^{1/2-j}/t$ and $c = \tau q^{1/2-j}$, therefore it is equal to

$$\frac{(q^{-j}, t\tau; q)_n}{(q^{1/2-j}\tau, q^{-1/2}t; q)_n}.$$

Consequently,

$$I_{mn}(t,\tau;q) = \sqrt{\pi}(q^{-1/2}\tau;q)_m e_q(t\tau q^n)E_q(-q^{1/2}t)E_q(-q^{1/2}\tau) \times \sum_{j=0}^{m} \frac{(q^{-m}, q^{1/2}/\tau; q)_j}{(q, q^{3/2-m}/\tau; q)_j} q^j \frac{(q^{-j}; q)_n}{(q^{1/2-j}\tau; q)_n}. \tag{A.8}$$

Due to the factor $(q^{-j};q)_n$ a sum over the index j in (A.8) is equal to zero for $n > m$, while for $n \leqslant m$ it starts with $j = n$. Therefore it can be written as

$$\sum_{j=0}^{m} \frac{(q^{-m}, q^{1/2}/\tau; q)_j}{(q, q^{3/2-m}/\tau; q)_j} q^j \frac{(q^{-j}; q)_n}{(q^{1/2-j}\tau; q)_n} = \frac{(q^{-m}; q)_n}{(q^{3/2-m}/\tau; q)_n} \left(\frac{q^{1/2}}{\tau}\right)^n \sum_{l=0}^{N} \frac{(q^{-N}, q^{1/2}/\tau; q)_l}{(q, q^{3/2-N}/\tau; q)_l} q^l \tag{A.9}$$

where $N = m - n$ and we have used the formulae (2.18) and (see, for example, [22])

$$(a;q)_{n+k} = (a;q)_n (aq^n; q)_k. \tag{A.10}$$

The remaining sum over l is calculated by the Chu–Vandermonde formula

$${}_2\phi_1(a, q^{-n}; c; q, q) = \frac{(c/a; q)_n}{(c; q)_n} a^n \tag{A.11}$$

and gives

$$\frac{(q^{1-N}; q)_N}{(q^{3/2-N}/\tau; q)_N} \left(\frac{q^{1/2}}{\tau}\right)^N = \delta_{N0} = \delta_{mn}$$

because of the evident relation $(q^{1-N}; q)_N = \delta_{N0}$, valid for any non-negative integer number N. Hence, the entire sum over j in (A.8) is equal to

$$\frac{(q^{-n}; q)_n}{(q^{3/2-n}/\tau; q)_n} \left(\frac{q^{1/2}}{\tau}\right)^n \delta_{mn} = \frac{(q;q)_n}{q^n (q^{-1/2}\tau; q)_n} \delta_{mn} \tag{A.12}$$

where we have again used (2.18) with $a = 1$ and $b = q^{3/2}/\tau$. Substituting (A.12) into (A.8), we obtain the following expression for the biorthogonality integral (A.1):

$$I_{mn}(t,\tau;q) = \sqrt{\pi}\frac{(q;q)_n}{q^n} e_q(q^n t\tau) E_q(-q^{1/2}t) E_q(-q^{1/2}\tau)\delta_{mn}$$

to complete the proof of (2.21).

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